Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 6238934, 7 pages
https://doi.org/10.1155/2020/6238934

Research Article

Generating Function for the Figurative Numbers of Regular Polyhedron

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Received 13 January 2020; Revised 21 March 2020; Accepted 26 March 2020; Published 14 April 2020

Academic Editor: Zhiyun Lin

In this paper, we are going to demonstrate a method for determining the generating functions of tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral figurative numbers. The method is based on the differences between the members of the series of the mentioned figurative numbers, as well as on the previously specified generating functions for the sequence $\sum_{n\geq 0}(n+1)x^n$ and geometric sequence $\sum_{n\geq 0}x^n$.

1. Introduction

The generating functions for polygonal figurative numbers (see [1–8]) as well as the generating functions for polyhedral figurative numbers (see [1, 9, 10]) have been the subject of research in the past period. Among polyhedral numbers, the authors of this paper find particularly interesting tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral figurative numbers. Their geometric representation is displayed by regular polyhedron: tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron (Figure 1).

Back in the 3rd century BC, Euclid proved that there exist only 5 regular polyhedrons (see [11, 12]). This is the reason why the polyhedron numbers are so special and that is why they deserve a special place in the set of figurative numbers.

The fact that octahedral and icosahedral numbers and their models exist in many scientific areas also contributes to this research. Icosahedral-hexagonal grid is the basis of the global numerical weather prediction model (GME). This grid was first introduced in meteorological modeling in 1968 and it has been gaining interest among researchers in recent years (see [13]). Icosahedral structures are also present in metals, such as gold (see [14]), copper (see [15]), and metal glasses (see [16]). Octahedral forms are present in virus structures (see [17, 18]), as well as in the atomic nucleus (see [19, 20]).

The procedure for determining the generating function of tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral numbers is based on the differences between the members of the series of objective numbers. The differences between the two adjacent figurative numbers, as well as the differences between these differences, provide great opportunities for determining many equivalents in the field of figurative numbers. Applying these principles, we are able to determine the generating functions of mentioned numbers which is the main result of this paper.

2. Materials and Methods

It is known that (see [1, 21])

Tetrahedral numbers: 1, 4, 10, 20, 35, 56, ...

Hexahedral numbers: 1, 8, 27, 64, 125, 216, ...

Octahedral numbers: 1, 6, 19, 44, 85, 146, ...

Dodecahedral numbers: 1, 20, 84, 220, 455, 816, ...

Icosahedral numbers: 1, 12, 48, 124, 255, 456, ...

We denote by Δ_1 the difference between two adjacent members in a series of figurative numbers, by Δ_2 the

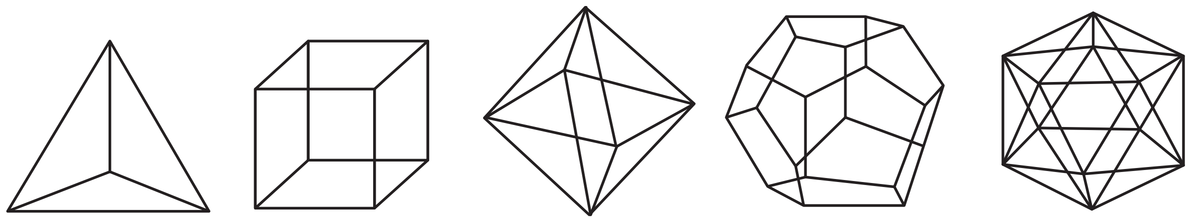

Figure 1: Tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron.

difference between two adjacent differences Δ_1, and by Δ_3 the difference between adjacent differences Δ_2.

Tetrahedral numbers: 1, 4, 10, 20, 35, 56, ...
Δ_1: 3, 6, 10, 15, 21, ...
Δ_2: 3, 4, 5, 6, ...
Δ_3: 1, 1, 1, ...

Hexahedral numbers: 1, 8, 27, 64, 125, 216, ...
Δ_1: 7, 19, 37, 61, 91, ...
Δ_2: 12, 18, 24, 30, ...
Δ_3: 6, 6, 6, ...

Octahedral numbers: 1, 6, 19, 44, 85, 146, ...
Δ_1: 5, 13, 25, 41, 61, ...
Δ_2: 8, 12, 16, 20, ...
Δ_3: 4, 4, 4, ...

Dodecahedral numbers: 1, 20, 84, 220, 455, 816, ...
Δ_1: 19, 64, 136, 235, 361, ...
Δ_2: 45, 72, 99, 126, ...
Δ_3: 27, 27, 27, ...

Icosahedral numbers: 1, 12, 48, 124, 255, 456, ...
Δ_1: 11, 36, 76, 131, 201, ...
Δ_2: 25, 40, 55, 70, ...
Δ_3: 15, 15, 15, ...

Figure 2 shows the formation of a series of tetrahedral numbers using the differences Δ_1. The second tetrahedral number 4 is created by adding the first difference $\Delta_1 = 3$ to the first tetrahedral number 1. Adding the following difference $\Delta_1 = 6$, a third tetrahedral number was formed $10 = 1 + 3 + 6$. Adding the following difference $\Delta_1 = 10$, we get the fourth tetrahedral number $20 = 1 + 3 + 6 + 10$, etc. Hexahedral, octahedral, dodecahedral, and icosahedral numbers are formed analogously.

The generating functions for polygonal figurative numbers (see [1]) are also known:

$$\text{triangular numbers:} \quad \frac{x}{(1-x)^3} = x + 3x^2 + 6x^3 + 10x^4 + \cdots, \quad \text{for } |x| < 1, \tag{1}$$

$$\text{square numbers:} \quad \frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots, \quad \text{for } |x| < 1, \tag{2}$$

$$\text{pentagonal numbers:} \quad \frac{x(1+2x)}{(1-x)^3} = x + 5x^2 + 12x^3 + 22x^4 + \cdots, \quad \text{for } |x| < 1, \tag{3}$$

$$\text{hexagonal numbers:} \quad \frac{x(1+3x)}{(1-x)^3} = x + 6x^2 + 15x^3 + 28x^4 + \cdots, \quad \text{for } |x| < 1. \tag{4}$$

The starting point for most generating functions (see [1]) is the geometric sequence:

$$\sum_{n\geq 0} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots, \quad |x| < 1, \tag{5}$$

which converges for $|x| < 1$. The generating function of geometric series is $(1/1 - x)$, i.e.,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots, \quad |x| < 1, \tag{6}$$

and it represents the sequence of ones, i.e., $1, 1, 1, \ldots, 1, \ldots$

The direct multiplication gives

$$\begin{aligned} \frac{1}{(1-x)^2} &= \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots\right) \\ &\quad \cdot \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots\right) \\ &= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots, \end{aligned} \tag{7}$$

and that is the generator function for sequence of natural numbers: $1, 2, 3, \ldots, n, \ldots$.

3. Results and Discussion

Denote by S1, S2, S3, S4, and S5 the sets of tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral figurative numbers, respectively.

Theorem 1. *The generating function for tetrahedral figurative numbers is*

$$f(S_1, x) = \frac{x}{(1-x)^4}, \quad \text{for } |x| < 1. \tag{8}$$

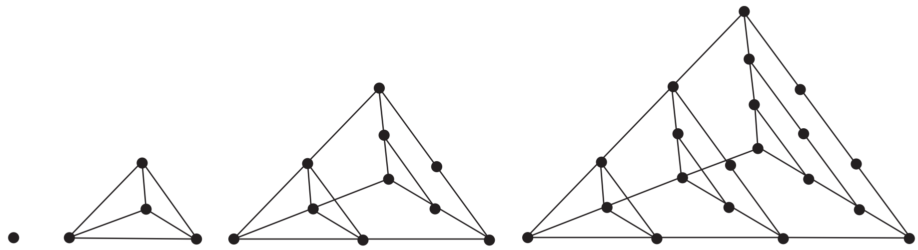

Figure 2: Formation of a series of tetrahedral numbers.

Proof. This theorem actually states that

$$\frac{x}{(1-x)^4} = x + 4x^2 + 10x^3 + 20x^4 + \cdots, \quad \text{for } |x| < 1. \tag{9}$$

To prove (9), we use the following notation:

$$A(x) = x + 4x^2 + 10x^3 + 20x^4 + \cdots. \tag{10}$$

We can rewrite previous expression as

$$A(x) = x \cdot \left(1 + 4x + 10x^2 + 20x^3 + \cdots\right). \tag{11}$$

By applying differences Δ_1 for tetrahedral numbers, the following holds:

$$\begin{aligned} A(x) &= x \cdot \left(1 + x + 3x + 4x^2 + 6x^2 + 10x^3 + 10x^3 + \cdots\right) \\ &= x \cdot \left(\left(1 + 3x + 6x^2 + 10x^3 + \cdots\right) + \left(x + 4x^2 + 10x^3 + \cdots\right)\right) \\ &= \left(x + 3x^2 + 6x^3 + 10x^4 + \cdots\right) + x \cdot A(x). \end{aligned} \tag{12}$$

Applying representation (1), we get

$$A(x) = \frac{x}{(1-x)^3} + x \cdot A(x). \tag{13}$$

Further on $A(x) - x \cdot A(x) = (x/(1-x)^3)$,

$$\begin{aligned} &\Longrightarrow A(x) \cdot (1-x) = \frac{x}{(1-x)^3}, \\ &\Longrightarrow A(x) = \frac{x}{(1-x)^4}, \end{aligned} \tag{14}$$

which was to be proved.

Taking the value $x = 0.01$, we get

$$A(0.01) = \frac{0.01}{(1-0.01)^4} = 0.(01)(04)(10)(20)(35)\ldots. \tag{15}$$

In the result obtained, separating by two decimal places, we obtain tetrahedral numbers: 1, 4, 10, ...

That is,

$$\begin{aligned} A(0.01) &= 0.01 + 4 \cdot 0.01^2 + 10 \cdot 0.01^3 + 20 \cdot 0.01^4 + \cdots \\ &= \sum_{n=1}^{\infty} 0.01^n \cdot \frac{n(n+1)(n+2)}{6}. \end{aligned} \tag{16}$$

Note that for value $x = 0.01$, tetrahedral numbers greater than 100 cannot be easily observed. In order to obtain better transparency, a value $x = 0.001$, or less, should be taken. This also holds in the following examples. □

Theorem 2. *The generating function for hexahedral figurative numbers is*

$$f(S_2, x) = \frac{x(x^2 + 4x + 1)}{(1-x)^4}, \quad \text{for } |x| < 1. \tag{17}$$

Proof. This theorem states that

$$\frac{x(x^2 + 4x + 1)}{(1-x)^4} = x + 8x^2 + 27x^3 + 64x^4 + \cdots, \quad \text{for } |x| < 1. \tag{18}$$

We prove this identity similarly as in the previous case.

Let $A(x) = x + 8x^2 + 27x^3 + 64x^4 + \cdots$

Then, $A(x) = x \cdot (1 + 8x + 27x^2 + 64x^3 + \cdots)$.

By applying the differences Δ_1 for hexahedral numbers, the following holds:

$$\begin{aligned} A(x) &= x \cdot \left(1 + x + 7x + 8x^2 + 19x^2 + 27x^3 + 37x^3 + \cdots\right) \\ &= x \cdot \left(\left(x + 8x^2 + 27x^3 + \cdots\right) + \left(1 + 7x + 19x^2 + 37x^3 + \cdots\right)\right) \\ &= x \cdot \left(A(x) + 1 + 7x + (7+12)x^2 + (7+12+18)x^3 + \cdots\right) \\ &= x \cdot \left(A(x) + 1 + 7 \cdot \left(x + x^2 + x^3 + \cdots\right) + 12 \cdot \left(x^2 + x^3 + \cdots\right) + 18 \cdot \left(x^3 + x^4 + \cdots\right) + \cdots\right) \\ &= x \cdot \left(A(x) + 1 + 7x \cdot \left(1 + x + x^2 + \cdots\right) + 6 \cdot 2x^2 \cdot \left(1 + x + x^2 + \cdots\right) + 6 \cdot 3x^3\left(1 + x + x^2 + \cdots\right) + \cdots\right). \end{aligned} \tag{19}$$

From the equality (6), we obtain

$$A(x) = x \cdot \Big(A(x) + 1 + 7x \cdot \frac{1}{1-x} + 6 \cdot 2x^2 \cdot \frac{1}{1-x} + 6 \cdot 3x^3 \cdot \frac{1}{1-x} + \cdots\Big)$$
$$= x \cdot A(x) + x \cdot \left(1 + \frac{7x}{1-x} + \frac{6x}{1-x} \cdot \left(2x + 3x^2 + 4x^3 + \cdots\right)\right). \tag{20}$$

Taking equality (7) leads to

$$A(x) - x \cdot A(x) = x \cdot \left(1 + \frac{7x}{1-x} + \frac{6x}{1-x} \cdot \left(\frac{1}{(1-x)^2} - 1\right)\right),$$
$$A(x) \cdot (1-x) = x \cdot \left(\frac{1-x+7x}{1-x} + \frac{6x}{1-x} \times \frac{2x - x^2}{(1-x)^2}\right). \tag{21}$$

By arranging this expression, we easily get the hexahedral figurative number generating function representation:

$$A(x) = \frac{x\left(x^2 + 4x + 1\right)}{(1-x)^4}. \tag{22}$$

Taking the value $x = 0.01$, we get

$$\mathrm{A}(0.01) = \frac{0.01 \cdot 1.0401}{(1-0.01)^4} = 0.(01)(08)(27)(64)\ldots. \tag{23}$$

In the result obtained, separating by two decimal places, we obtain hexahedral numbers: 1, 8, 27, 64, ... □

Theorem 3. *The generating function for octahedral figurative numbers is*

$$f(S_3, x) = \frac{x(x+1)^2}{(1-x)^4}, \quad \text{for } |x| < 1. \tag{24}$$

Proof. We need to prove that

$$\frac{x(x+1)^2}{(1-x)^4} = x + 6x^2 + 19x^3 + 44x^4 + 85x^5 + \cdots, \quad \text{for } |x| < 1. \tag{25}$$

For $A(x) = x + 6x^2 + 19x^3 + 44x^4 + 85x^5 + \cdots$, the next is valid:

$$\begin{aligned}
A(x) &= x \cdot \left(1 + 6x + 19x^2 + 44x^3 + 85x^4 + \cdots\right)\\
&= x \cdot \left(1 + (x + 5x) + \left(6x^2 + 13x^2\right) + \left(19x^3 + 25x^3\right) + \left(44x^4 + 41x^4\right) + \cdots\right)\\
&= x \cdot \left(1 + 6x^2 + 19x^3 + 44x^4 + \cdots + 1 + 5x + 13x^2 + 25x^3 + 41x^4 + \cdots\right)\\
&= x \cdot \left(A(x) + 1 + 5x + 5x^2 + 8x^2 + 5x^3 + 8x^3 + 12x^3 + 5x^4 + 8x^4 + 12x^4 + 16x^4 + \cdots\right)\\
&= x \cdot \left(A(x) + 1 + 5x + 5x^2 + 5x^3 + 5x^4 + \cdots + 8x^2 + 8x^3 + 8x^4 + \cdots + 12x^3 + 12x^4 + \cdots\right)\\
&= x \cdot \left(A(x) + 1 + 5 \cdot \left(x + x + x^2 + x^3 + \cdots\right) + 8x \cdot \left(x + x^2 + x^3 + \cdots\right) + 12x^2 \cdot \left(x + x^2 + x^3 \cdots\right) + \cdots\right)\\
&= x \cdot \left(A(x) + 1 + 5 \cdot \left(\frac{1}{1-x} - 1\right) + 8x \cdot \left(\frac{1}{1-x} - 1\right) + 12^2 \cdot \left(\frac{1}{1-x} - 1\right) + \cdots\right)\\
&= x \cdot \left(A(x) + 1 + \frac{x}{1-x} \cdot \left(5 + 8x + 12x^2 + \cdots\right)\right)\\
&= x \cdot \left(A(x) + 1 + \frac{5x}{1-x} + \frac{x}{1-x} \cdot \left(4 \cdot 2x + 4 \cdot 3x^2 + 4 \cdot 4x^3 \cdots\right)\right)\\
&= x \cdot A(x) + x + \frac{5x^2}{1-x} + \frac{4x^2}{1-x} \cdot \left(2x + 3x^2 + 4x^3 \cdots\right).
\end{aligned} \tag{26}$$

From the equality (7), we obtain

$$A(x) - x \cdot A(x) = x + \frac{5x^2}{1-x} + \frac{4x^2}{1-x} \cdot \left(\frac{1}{(1-x)^2 - 1}\right). \quad (27)$$

By arranging this expression, we get that

$$A(x) \cdot (1-x) = \frac{x + 2x^2 + x^3}{(1-x)^3},$$
$$\Longrightarrow A(x) = \frac{x(1+x)^2}{(1-x)^4}, \quad (28)$$

which was to be proved.

Taking the value $x = 0.01$, we get

$$A(0.01) = \frac{0.01 \cdot 1.0201}{(1-0.01)^4} = 0.(01)(06)(19)(44)\ldots. \quad (29)$$

Separating by two decimal places, we obtain octahedral numbers: $1, 6, 9, 44, \ldots$. □

Theorem 4. *The generating function for dodecahedral figurative numbers is*

$$f(S_4, x) = \frac{x(10x^2 + 16x + 1)}{(1-x)^4}, \quad \text{for } |x| < 1. \quad (30)$$

Proof. We will prove this theorem by confirming that

$$\frac{x(10x^2 + 16x + 1)}{(1-x)^4} = x + 20x^2 + 84x^3 + 220x^4 + 455x^5 + \cdots, \quad \text{for } |x| < 1. \quad (31)$$

We denote by $A(x) = x + 20x^2 + 84x^3 + 220x^4 + 455x^2 + \cdots$

Then,

$$\begin{aligned}
A(x) &= x \cdot \left(1 + 20x + 84x^2 + 220x^3 + 455x^4 + \cdots\right) \\
&= x \cdot \left(1 + x + 19x + 20x^2 + 64x^2 + 84x^3 + 136x^3 + 220x^4 + 235x^4 + \cdots\right) \\
&= x \cdot \left(x + 20x^2 + 84x^3 + 220x^4 + \cdots + 1 + 19x + 64x^2 + 136x^3 + 235x^4 + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 19x \cdot (19+45)x^2 + (19+45+72)x^3 + (19+45+72+99)x^4 + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 19x \cdot \left(1 + x + x^2 + \cdots\right) + 45x^2 \cdot \left(1 + x + x^2 + \cdots\right) + 72x^3 \cdot \left(1 + x + x^2 + \cdots\right) + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 19x \cdot \frac{1}{1-x} + 45x^2 \cdot \frac{1}{1-x} + 72x^3 \cdot \frac{1}{1-x} + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + \frac{x}{1-x} \cdot \left(19 + 45x + 72x^2 + 99x^3 + \cdots\right)\right) \\
&= x \cdot A(x) + x + \frac{x^2}{1-x} \cdot 19 + \frac{x^2}{1-x} \cdot 9 \cdot \left(5x + 8x^2 + 11x^3 + \cdots\right) \\
&= x \cdot A(x) + x + \frac{19x^2}{1-x} + \frac{9x^2}{1-x} \cdot \left(5x + 5x^2 + 3x^2 + 5x^3 + 2 \cdot 3x^3 + \cdots\right)
\end{aligned}$$
$$A(x) - x \cdot A(x) = x + \frac{19x^2}{1-x} + \frac{9x^2}{1-x} \cdot \left(5x \cdot \left(1 + x + x^2 + x^3 + \cdots\right) + 3x^2 \cdot \left(1 + 2x + 3x^2 + 4x^3 + \cdots\right)\right). \quad (32)$$

From the equality (6) and (7), we obtain

$$A(x) \cdot (1-x) = x + \frac{19x^2}{1-x} + \frac{9x^2}{1-x} \cdot \left(5x \cdot \frac{1}{1-x} + 3x^2 \cdot \frac{1}{(1-x)^2}\right), \quad (33)$$

and after arranging this expression, we get that

$$A(x) \cdot (1-x) = \frac{10x^3 + 16x^2 + x}{(1-x)^3},$$
$$\Longrightarrow A(x) = \frac{x(10x^2 + 16x + 1)}{(1-x)^4}, \quad (34)$$

which was to be proved.

Taking the value $x = 0.001$, we get

$$A(0.0001) = \frac{0.001 \cdot 1.01601}{(1 - 0.001)^4} = 0.(001)(020)(084)(220)\ldots. \quad (35)$$

Separating by 3 decimal places, we obtain dodecahedral numbers: 1, 20, 84, ... □

Theorem 5. *The generating function for icosahedral figurative numbers is*

$$f(S_5, x) = \frac{x(6x^2 + 8x + 1)}{(1 - x)^4}, \quad \text{for } |x| < 1. \quad (36)$$

Proof. Let us show that the following identity is true:

$$\frac{x(6x^2 + 8x + 1)}{(1 - x)^4} = x + 12x^2 + 48x^3 + 124x^4 + 225x^5 + \cdots, \quad \text{for } |x| < 1. \quad (37)$$

Then,

$$\begin{aligned}
A(x) &= x + 12x^2 + 48x^3 + 124x^4 + 255x^5 + \cdots \\
&= x \cdot \left(1 + 12x + 48x^2 + 124x^3 + 225x^4 + \cdots\right) \\
&= x \cdot \left(1 + x + 11x + 12x^2 + 36x^2 + 48x^3 + 76x^3 + 124x^4 + 131x^4 + \cdots\right) \\
&= x \cdot \left(x + 12x^2 + 48x^3 + 124x^4 + \cdots + 1 + 11x + 36x^2 + 76x^3 + 131x^4 + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 11x + (11 + 25)x^2 + (11 + 25 + 40)x^3 + (11 + 25 + 40 + 55)x^4 + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 11x \cdot \left(1 + x + x^2 + \cdots\right) + 25x^2 \cdot \left(1 + x + x^2 + \cdots\right) + 40x^3 \cdot \left(1 + x + x^2 + \cdots\right) + \cdots\right) \\
&= x \cdot \left(A(x) + 1 + 11x \cdot \frac{1}{1 - x} + 25x^2 \cdot \frac{1}{1 - x} + 40x^3 \cdot \frac{1}{1 - x} + \cdots\right) \\
&= x \cdot A(x) + x + \frac{1}{1 - x} \cdot \left(11x^2 + 25x^3 + 40x^4 + 55x^5 + \cdots\right) \\
&= x \cdot A(x) + x + \frac{1}{1 - x} \cdot \left(11x^2 + 25x^3 + 25x^4 + 15x^4 + 25x^5 + 2.15x^5 + \cdots\right) \\
&= x \cdot A(x) + x + \frac{1}{1 - x} \cdot \left(11x^2 + 25x^3 + 25x^4 + 25x^4 + 25x^5 + \cdots + 15x^4 + 2 \times 15x^5 + \cdots\right) \\
&= x \cdot A(x) + x + \frac{1}{1 - x} \cdot \left(11x^2 + 25x^3 \cdot \left(1 + x + x^2 + \cdots\right) + 15x^4 \cdot \left(1 + 2x + 3x^2 + \cdots\right)\right) \\
A(x) - x \cdot A(x) &= x + \frac{1}{1 - x} \times \left(11x^2 + 25x^3 \cdot \frac{1}{1 - x} + 15x^4 \cdot \frac{1}{(1 - x)^2}\right) \\
A(x) \cdot (1 - x) &= \frac{x(6x^2 + 8x + 1)}{(1 - x)^3}, \\
\Longrightarrow A(x) &= \frac{x(6x^2 + 8x + 1)}{(1 - x)^4},
\end{aligned} \quad (38)$$

which was to be proved.

Taking the value $x = 0.001$, we get

$$A(0.001) = \frac{0.001 \cdot 1.008006}{(1 - 0.001)^4} = 0.(001)(012)(048)(124)\ldots. \quad (39)$$

Separating by 3 decimal places, we obtain icosahedral numbers: 1, 12, 48, ...

The obtained results can be reached in a different way. The authors of this paper chose the presented method because of its simplicity and obviousness. □

4. Main Text

Polyhedron figurative numbers with their models exist in many scientific fields. In this paper, we presented a procedure for determining the generating function of tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral figurative numbers.

5. Conclusion

In this paper, we determined tetrahedral, hexahedral, octahedral, dodecahedral, and icosahedral generating functions' representation:

$$\begin{aligned} f(S_1,x) &= \frac{x}{(1-x)^4}, \quad \text{for } |x|<1, \\ f(S_2,x) &= \frac{x(x^2+4x+1)}{(1-x)^4}, \quad \text{for } |x|<1, \\ f(S_3,x) &= \frac{x(x+1)^2}{(1-x)^4}, \quad \text{for } |x|<1, \\ f(S_4,x) &= \frac{x(10x^2+16x+1)}{(1-x)^4}, \quad \text{for } |x|<1, \\ f(S_5,x) & \frac{x(6x^2+8x+1)}{(1-x)^4}, \quad \text{for } |x|<1. \end{aligned} \tag{40}$$

Applying the generating functions, we can generate strings of appropriate figurative numbers and apply them in further studies.

Data Availability

The data used to support the conclusions of the study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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