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KNOT BENDING

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Abstract. In this paper we point out the geometric aspect of knots. By the term knot we mean a closed self-avoiding curve in a 3-dimensional Euclidean space. We consider infinitesimal bending of knots and examine the behavior of torus knots under this type of deformation.

1. INTRODUCTION

A knot is a closed, self-avoiding curve. From the topological point of view, we have more general definition. In order to give a topological definition, some basic terms will be introduced.

A **homeomorphism** between two topological spaces is continuous bijection $h: X \to Y$ whose inverse is also continuous. If such a map exists, then X and Y are **homeomorphic** or **topology equivalent**.

Let be given a smooth closed curve C in $\mathbb{R}^3$. The **homeomorphic** curves C, C' are called **isotopic** if there exists a continuous family of curves C_t depending on t, $(0 \leq t \leq 1)$, such that C_t is homeomorphic to C, $C_0 = C$ and $C_1 = C'$. We say that C is a **knot** if it is homeomorphic to a circle but is not isotopic to a circle.

According to this definition, another knot C' is equivalent to C if it can be continuously deformed into C without crossing itself during the process. Equivalent knots are considered the same in the topological sense and determine an equivalence class of knots named **knot type**. It is therefore possible to think of a knot as a curve with small but positive thickness which allows us to present it as a tube. But in geometrical sense we can observe a particular representative of a knot type as a closed self-avoiding curve in 3-dimensional space.

The simplest knot is the **unknot** also known as the trivial knot which can be deformed to a geometric circle in $\mathbb{R}^3$. Two other rather easy knots are the **trefoil** and the **figure-eight knots**.

In classical knot theory, mathematicians are often interested in knot classes and how to distinguish between them. Contrary to this approach, geometric knot theory deals with the specific shape of knots and how to find or compute particularly nice representatives of a given knot class. In this sense knot theory

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studies relations between the geometry of space curves and the knot types they represent. Therefore, in geometric sense we could observe small deformations of knots as the family of different curves.

Knots are interesting not only from a mathematical point of view, but also from the aspect of other sciences such as physics, chemistry, biology, computer graphics, etc. (see [2], [3], [5], [10]).

The study of deformations dates from the ancient times and stems from purely practical problems. In physics, a deformation means a change of a shape of body under the influence of the external or internal forces. In the case when the body returns to its original shape after the termination of the effect of forces, it is said to be elastic, otherwise, if it is deformed permanently, we say it is plastic. Testing of deformations of different materials is of great importance in civil engineering. Measurement of deformations of high buildings allows us to provide an adequate level of safety and security from potential damage and disaster. Deformations are in close connection with thin elastic shell and have a huge application from the mechanical point of view. In biology, the notion of deformation has also found his place. We are talking about the elasticity of the cell membrane in connection with the fluidity that allows the proteins to move along the membrane.

In geometry, the problem of deformations is covered by so-called the **surface bending theory**. The surface bending theory considers the bending of surfaces, ie. the isometric deformations, as well as the infinitesimal bending of surfaces. Under bending, surface is included in continuous family of isometric surfaces, so that any curve on the surface preserves its arc length. The angles are also preserved. On the other hand, **infinitesimal bending** of surfaces is not an isometric deformation, or roughly speaking it is an isometric deformation with appropriate precision. Arc length is stationary under infinitesimal bending with a given precision. In the case of infinitesimal bending the surface is deformed so that in the initial moment of a deformation, the arc length on the surface is stationary, i. e. initial velocity of its change is zero.

In addition to infinitesimal bending of surfaces, infinitesimal bending of curves and manifolds is also considered in bending theory.

In the present article we consider infinitesimal bending of a knot as a closed simple curve. Some papers related to this topic are [4], [6], [8], [9].

2. INFINITESIMAL BENDING

Let us consider a regular curve

$$C: \mathbf{r} = \mathbf{r}(u), \ \ u \in J \subseteq \mathbb{R}^3 \tag{1}$$

of a class C^k, $k \geq 3$, included in a family of the curves

$$C_\epsilon: \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}^{(1)}(u) + \epsilon^2 \mathbf{z}^{(2)}(u) + \cdots + \epsilon^m \mathbf{z}^{(m)}(u), \ \ m \geq 1, \tag{2}$$

where $\epsilon \geq 0, \epsilon \to 0$, and we get C for $\epsilon = 0$ ($C = C_0$). The fields $\mathbf{z}^{(j)}(u) \in C^k, k \geq 3, j = 1, \dots, m,$ are vector functions defined in the points of C.

Definition 1. [1] *A family of curves* C_ϵ (2) *is an* **infinitesimal bending of the order** m **of the curve** C (1) *if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon^m). \tag{3}$$

The field $\mathbf{z}^{(j)} = \mathbf{z}^{(j)}(u)$ *is the* **infinitesimal bending field of the order** j, $j = 1, \dots, m$, *of the curve* C.

Based on the definition of the infinitesimal bending of the order m and Eq. (3), the next relation is valid

$$d\mathbf{r}_\epsilon^2 - d\mathbf{r}^2 = (d\mathbf{r}_\epsilon - d\mathbf{r}) \cdot (d\mathbf{r}_\epsilon + d\mathbf{r}) = o(\epsilon^m).$$

From here we have

$$d\left(\sum_{j=1}^{m} \epsilon^j \mathbf{z}^{(j)}\right) \cdot d\left(2\mathbf{r} + \sum_{j=1}^{m} \epsilon^j \mathbf{z}^{(j)}\right) = o(\epsilon^m),$$

or more precise

$$\left(\epsilon d\mathbf{z}^{(1)} + \epsilon^2 d\mathbf{z}^{(2)} + \cdots + \epsilon^m d\mathbf{z}^{(m)}\right) \cdot \left(2d\mathbf{r} + \epsilon d\mathbf{z}^{(1)} + \epsilon^2 d\mathbf{z}^{(2)} + \cdots + \epsilon^m d\mathbf{z}^{(m)}\right) = o(\epsilon^m).$$

The necessary and sufficient condition for the left side to be infinitesimal value with respect to ϵ^m is to be

$$d\mathbf{r} \cdot d\mathbf{z}^{(1)} = 0,\ 2d\mathbf{r} \cdot d\mathbf{z}^{(j)} + \sum_{l=1}^{j-1} d\mathbf{z}^{(l)} \cdot d\mathbf{z}^{(j-l)} = 0,\ j = 2, \dots, m. \tag{4}$$

According to that, the next theorem states.

Theorem 1. [1] *Necessary and sufficient condition for the curves* C_ϵ, (2), *to be infinitesimal bending of the* m *−th order of the curve* C, (1), *is to be valid* (4).

If infinitesimal bending is reduced to rigid motion of the curve, without internal deformations, we say it is **trivial** infinitesimal bending. The corresponding bending field is also called trivial.

Specially, infinitesimal bending of the first order, or shorter, infinitesimal bending, is a family of curves

$$C_\epsilon : \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}(u),$$

where $\mathbf{z}(u)$ is an infinitesimal bending field (of the first order).

The following theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 2. [12] *The infinitesimal bending field for the curve* C*is*

$$\mathbf{z}(u) = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\, du, \tag{5}$$

where $p(u)$ *and* $q(u)$ *are arbitrary integrable functions, and vectors* $\mathbf{n}_1(u)$ *and* $\mathbf{n}_2(u)$ *are respectively unit principal normal and binormal vector fields of a curve* C.

Similarly, for the infinitesimal bending of the second order we have the following theorem.

Theorem 3. [7] The infinitesimal bending fields of the first and the second order of the curve C are respectively

$$\mathbf{z}^{(1)} = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\, du,$$

$$\mathbf{z}^{(2)} = \int \left[-\frac{p^2(u) + q^2(u)}{2\|\dot{\mathbf{r}}\|}\mathbf{t} + r(u)\mathbf{n}_1 + g(u)\mathbf{n}_2 \right] du,$$

where $p(u), q(u), r(u), g(u)$ are arbitrary integrable functions and vectors $\mathbf{t}(u)$, $\mathbf{n}_1(u)$, $\mathbf{n}_2(u)$ are unit tangent, principal normal and binormal vector fields, respectively, of the curve C.

Under infinitesimal bending, geometric magnitudes of the curve are changed which is described with variations of these geometric magnitudes.

Definition 2. [11] Let $A = A(u)$ be a magnitude which characterizes a geometric property on the curve C and $A_\epsilon(u)$ the corresponding magnitude on the curve C_ϵ being infinitesimal bending of the curve C, and set

$$\Delta A = A_\epsilon - A = \epsilon\delta A + \epsilon^2\delta^2 A + \cdots + \epsilon^n\delta^n A + \cdots$$

The coefficients δA, $\delta^2 A, \dots, \delta^n A$ are **the first, the second**, ..., **the n-th variation** of the geometric magnitude A, respectively, under the infinitesimal bending C_ϵ of the curve C.

3. INFINITESIMAL BENDING OF KNOTS

In geometric sense, small deformations of a knot (a particular curve from the corresponding knot type) can be considered as the family of different curves. We are going to demonstrate this fact through examples.

Let us consider the trefoil knot

$$\mathbf{r}(u) = (4\cos 2u + 2\cos u, 4\sin 2u - 2\sin u, \sin 3u).$$

The field

$$\mathbf{z} = \int \frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2} du = (60\cos u - 6\cos 2u - \frac{3}{2}\cos 4u + \frac{12}{5}\cos 5u,$$

$$-60\sin u - 6\sin 2u + \frac{3}{2}\sin 4u + \frac{12}{5}\sin 5u, 124u - \frac{16}{3}\sin 3u),$$

$u \in [0, 2\pi]$, is corresponding infinitesimal bending field. This field is obtained, according to (5), for $p(u) = 0$ and $q(u) = \left\| \frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2} \right\|$. Obviously, $\mathbf{z}(0) \neq \mathbf{z}(2\pi)$, and the knot "is torn" under this infinitesimal bending, see Figs. 1.

If we want to get a family of closed curves under infinitesimal bending of a knot, we must specify the condition $\mathbf{z}(0) = \mathbf{z}(2\pi)$ for the infinitesimal bending field.

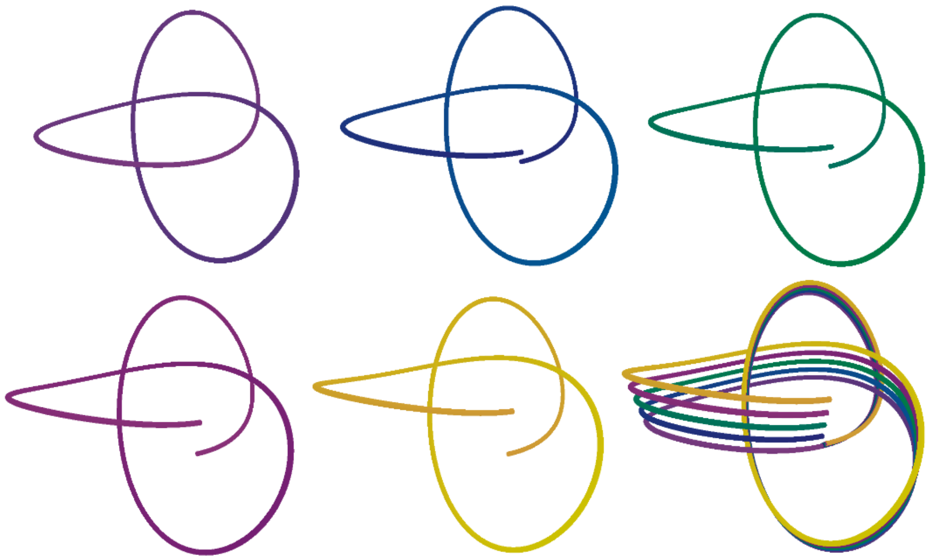

Figure 1: The trefoil knot and its infinitesimal bending for $\epsilon = 0.0005, 0.001, 0.0015, 0.002$.

For $p(u) = \left\|\left(\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2}\right) \times \frac{d\mathbf{r}}{du}\right\|$, $q(u) = 0$, we obtain

$$\mathbf{z} = \int \left(\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2}\right) \times \frac{d\mathbf{r}}{du} du$$

$$= (303 \sin u - 540 \sin 2u - \frac{53}{2} \sin 4u + \frac{46}{5} \sin 5u + \frac{9}{7} \sin 7u + \frac{9}{4} \sin 8u, 303 \cos u + 540 \cos 2u - \frac{53}{2} \cos 4u - \frac{46}{5} \cos 5u + \frac{9}{7} \cos 7u - \frac{9}{4} \cos 8u\,, -12(-17 \cos 3u + \cos 6u)).$$

All bent curves are also closed, see Figs. 2.

4. Frenet-Serret frame under infinitesimal bending

Let us consider an infinitesimal bending of the second order of the knot $C: \mathbf{r} = \mathbf{r}(s) = \mathbf{r}(u(s)), s \in I \subseteq \mathbb{R}$, parameterized by arc length s:

$$C_\epsilon: \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon \mathbf{z}^{(1)}(s) + \epsilon^2 \mathbf{z}^{(2)}(s).$$

Since the vector fields $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$ are defined in the points of C, they can be presented in the form

$$\mathbf{z}^{(j)} = z^{(j)}\mathbf{t} + z_1^{(j)}\mathbf{n}_1 + z_2^{(j)}\mathbf{n}_2, \;\; j = 1,2,$$

where $z^{(j)}\mathbf{t}$ is tangential and $z_1^{(j)}\mathbf{n}_1 + z_2^{(j)}\mathbf{n}_2$ is normal component, $z^{(j)}$, $z_1^{(j)}$, $z_2^{(j)}$ are the functions of s; $\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2$ are unit vectors of Frenet-Serret frame which form an orthonormal basis spanning $\mathbb{R}^3$.

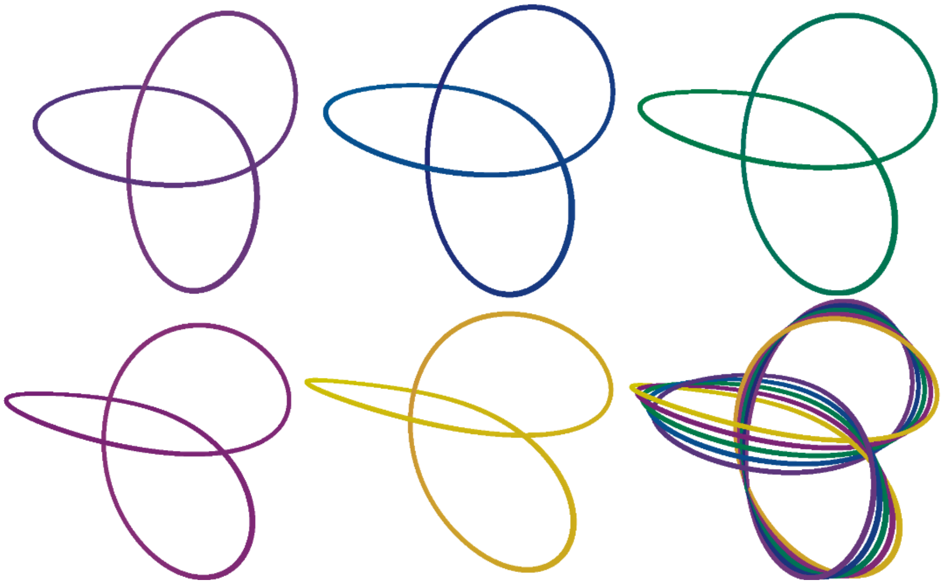

Figure 2: The trefoil knot and its infinitesimal bending for $\epsilon = 0.00025, 0.0005, 0.00075, 0.001$.

Under infinitesimal bending of the second order of knots, unit vectors of Frenet-Serret frame make changes which is described by their variations of the first and the second order. Most of them have already been determined. Thus, $\delta\mathbf{t}, \delta\mathbf{n}_1, \delta\mathbf{n}_2, \delta^2\mathbf{t}$ are obtained in [7], and $\delta^2\mathbf{n}_1$ in [8]. It remains to examine $\delta^2\mathbf{n}_2$, which we will do below.

Theorem 4. Under second order infinitesimal bending of the knot C, the second variation of the binormal vector is

$$\delta^2\mathbf{n}_2 = f_t\mathbf{t} + f_{n_1}\mathbf{n}_1 + f_{n_2}\mathbf{n}_2,$$

where

$$f_t = -\left(z_2^{(2)'} + \tau z_1^{(2)}\right) + \frac{1}{k}\left(kz^{(1)} + z_1^{(1)'} - \tau z_2^{(1)}\right)\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right),$$

$$f_{n_1} = \frac{1}{\tau}\Big\{kg_t + g'_{n_1} - \tau g_{n_2} + k\left(kz^{(2)} + z_1^{(2)'} - \tau z_2^{(2)}\right) + \left(k'z^{(1)} + z_1^{(1)''} + (k^2-\tau^2)z_1^{(1)} - 2\tau z_2^{(1)'} - \tau' z_2^{(1)}\right)\left(kz^{(1)} + z_1^{(1)'} - \tau z_2^{(1)}\right) + \frac{1}{k}\left[\tau' z^{(1)} + 2k\tau z_1^{(1)} + kz_2^{(1)'} + \left(\frac{1}{k}(2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)})\right)'\right]\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right)\Big\},$$

$$f_{n_2} = -\frac{1}{2}\left[\left(z_2^{(1)'} + \tau z_1^{(1)}\right)^2 + \frac{1}{k^2}\left(k\tau z^{(1)} + 2\tau z_1^{(1)'} + \tau' z_1^{(1)} + z_2^{(1)''} - \tau^2 z_2^{(1)}\right)^2\right],$$

k is the curvature, τ is the torsion, g_t, g_{n_1}, g_{n_2} are the tangent, the normal and the binormal component of $\delta^2\mathbf{n}_1$, respectively.

Proof. Applying the second variation of the equation $\mathbf{n}_2 \cdot \mathbf{t} = 0$, we obtain

$$\mathbf{t} \cdot \delta^2\mathbf{n}_2 = -\mathbf{n}_2 \cdot \delta^2\mathbf{t} - \delta\mathbf{n}_2 \cdot \delta\mathbf{t}.$$

We used here the following property of the second variation:

$$\delta^2 AB = A\delta^2 B + B\delta^2 A + \delta A\delta B,$$

whereby A and B are some magnitudes which characterize some geometric properties of the knot. If we use the expressions for $\delta\mathbf{t}$, $\delta^2\mathbf{t}$ and $\delta\mathbf{n}_2$ from [7], we obtain $f_t = \mathbf{t} \cdot \delta^2\mathbf{n}_2$.

Further, if we take the second variation of the well-known second Frenet-Serret equation $\mathbf{n}_1' = -k\mathbf{t} + \tau\mathbf{n}_2$ and dot with $\mathbf{n}_1$, we obtain

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_2 = \frac{1}{\tau}(\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1{}' + k\mathbf{n}_1 \cdot \delta^2\mathbf{t} + \delta k\, \mathbf{n}_1 \cdot \delta\mathbf{t} - \delta\tau\mathbf{n}_1 \cdot \delta\mathbf{n}_2).$$

Since $\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1' = \mathbf{n}_1 \cdot (\delta^2\mathbf{n}_1)'$, after using the Frenet-Serret equations we obtain

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1' = kg_t + g_{n_1}' - \tau g_{n_2},$$

where g_t, g_{n_1}, g_{n_2} are the tangent, the normal and the binormal component of $\delta^2\mathbf{n}_1$, respectively. Also, using the expressions for δk and $\delta\tau$ [7], we obtain $\mathbf{n}_1 \cdot \delta^2\mathbf{n}_2 = f_{n_1}$.

Finally, we will determine the binormal component of $\delta^2\mathbf{n}_2$. Using the second variation of the equation $\mathbf{n}_2 \cdot \mathbf{n}_2 = 1$, we obtain

$$\mathbf{n}_2 \cdot \delta^2\mathbf{n}_2 = -\frac{1}{2}\delta\mathbf{n}_2 \cdot \delta\mathbf{n}_2.$$

Applying $\delta\mathbf{n}_2$ from [7] we obtain $f_{n_2} = \mathbf{n}_2 \cdot \delta^2\mathbf{n}_2$. □

5. TORUS KNOT BENDING

A **torus knot** is a special kind of knots that lies on the surface of a torus. The parametric equations of (p, q)-torus knot are:

$$x(t) = (c + a\cos pt)\cos qt,$$
$$y(t) = (c + a\cos pt)\sin qt, \qquad (6)$$
$$z(t) = a\sin pt,$$

$t \in [0,2\pi)$, $a, c \in \mathbb{R}$. The parametric equations of a torus are:

$$x(u, v) = (c + a\cos v)\cos u,$$
$$y(u, v) = (c + a\cos v)\sin u, \qquad (7)$$
$$z(u, v) = a\sin v,$$

whereby the c is the distance from the center of the tube to the center of the torus (major radius), and the a is the radius of the tube (minor radius).

Let us determine an infinitesimal bending field so that all bent torus knots are on the same surface of the torus with a given precision, i.e.

$$F\big(x(t), y(t), z(t)\big) = 0,$$
$$F(x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) = o(\epsilon),$$

where $F(x, y, z) = 0$ is the implicit torus equation.

Theorem 5. *Infinitesimal bending field* $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ *satisfying the condition*

$$z_3(t) = -(z_1(t)\cos qt + z_2(t)\sin qt)\cot pt \tag{8}$$

includes the torus knot (6) *under infinitesimal bending of the first order into the family of deformed curves on the torus* (7).

Proof. An implicit equation in Cartesian coordinates for a torus radially symmetric about the z-axis is

$$\left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 = a^2, \tag{9}$$

or the solution of $F(x, y, z) = 0$, where

$$F(x, y, z) = \left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 - a^2.$$

Let C be a torus knot that bends infinitesimally so that all deformed curves are on the same torus with a given precision and let (6) be its parametric equation. The family of deformed knots C_ϵ under infinitesimal bending is

$$C_\epsilon : \begin{cases} (c + a\cos pt)\cos qt + \epsilon z_1(t) \\ (c + a\cos pt)\sin qt + \epsilon z_2(t) \\ a\sin pt + \epsilon z_3(t). \end{cases}$$

Since the curves C_ϵ should be on the torus (7) ie. (9), it must be valid

$$\left(c - \sqrt{((c + a\cos pt)\cos qt + \epsilon z_1(t))^2 + ((c + a\cos pt)\sin qt + \epsilon z_2(t))^2}\right)^2 + (a\sin pt + \epsilon z_3(t))^2 = a^2,$$

after neglecting the terms of order higher than 1with respect to ϵ. From the last equation, after a little calculation, we obtain the condition (8) for the infinitesimal bending field. □

By examining the condition (8) we check whether the deformed knot remains on the torus. Also, using this condition we can determine the infinitesimal bending field $\mathbf{z}$ taking into account the condition $d\mathbf{r} \cdot d\mathbf{z} = 0 \Leftrightarrow \dot{\mathbf{r}}(t) \cdot \dot{\mathbf{z}}(t) = 0$. Thus, we obtain the infinitesimal bending field under which all deformed curves are on the torus with a given precision.

6. Conclusions

The geometric knot theory includes the consideration of knots as space curves which allows us to consider them from the point of view of geometry. In this regard it is possible to distinguish different representatives from the same knot type as an equivalence class. In this paper we pointed out to this fact considering the infinitesimal bending of particular representatives of knots. We examined the change of the unit vectors of Frenet-Serret frame under this type of deformation. Finally, we investigated the special kind of knots, so called torus knot, and its infinitesimal bending on the torus.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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